# October Math Gems

PROBLEM OF THE WEEK 14

## §1 Problems

**Problem 1.1.** The sum of integers from 1 to 100 that are divisible by 2 or 5 is

*Solution.* The result= sum of integers divisible by 2+ sum of integers divisible by 5 - sum of integers divisible by 10
(2+4+6....+100)+(5+10+15....100)-(10+20+30.....100)=

$\frac{50}{2}(2+100)+\frac{20}{2}(5+100)-\frac{10}{2}(10+100)=2550+1050-550=3050$ □

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**Problem 1.2.** If $f(x)=\cos(\log x)$, then $f(\frac{1}{x})f(\frac{1}{y})-\frac{1}{2}(f(\frac{x}{y})+f(xy))=$

*Solution.* $f(\frac{1}{x})f(\frac{1}{y})-\frac{1}{2}(f(\frac{x}{y})+f(xy))=$

$\cos\log(\frac{1}{x})*\cos\log(\frac{1}{y})-(\cos\log(\frac{x}{y})+\cos\log(xy))=$

$\cos\log(\frac{1}{x})*\cos\log(\frac{1}{y})-\cos\frac{\log(\frac{x}{y})+\log(xy)}{2}\cos\frac{\log(\frac{x}{y})-\log(xy)}{2}=$

$\cos\log(\frac{1}{x})*\cos\log(\frac{1}{y})-\cos(\frac{1}{2}\log x^2)\cos(\frac{1}{2}\log\frac{1}{y^2})=$

$\cos(-\log x)\cos(-\log y)-\cos(\log x)\cos(-\log y)=0$ □

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**Problem 1.3.** If $f(x)=x^2-1, g(x)=2x+3$ then $fogof(1)=$

*Solution.* $fogof(1)=f(g(1^2-1)=f(2(0)+3)=f(3)=3^2-1=8$ □

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**Problem 1.4.** $\log x+\log x^3+\log x^5....+\log x^{2n-1}=$

*Solution.* $\log x+\log x^3+\log x^5....+\log x^{2n-1}=$

$\log(x*x^3*x^5....x^{2n-1})=\log x^{1+3+5+.....+2n-1}=\log x^{n^2}=n^2\log x$ □

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**Problem 1.5.** If $\log 2+\frac{1}{2}\log a+\frac{1}{2}\log b=\log(a+b)$ then find the value of $a,b$

*Solution.* $2\log 2+\log a+\log b=2\log(a+b)$
$\log(4ab)=\log(a+b)^2$
$4ab=(a+b)^2,(a-b)^2=0,a=b$ □

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**Problem 1.6.** If $f(x+1)=x^2-3x+2$ then $f(x)=$

*Solution.* We have that $f(x+1) = x^2 - 3x + 2$, $f(x) = (x-1)^2 - 3(x-1) + 2$, then $f(x) = x^2 - 5x + 6$ □

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**Problem 1.7.** If $f(x) = \frac{3x+2}{5x-3}$, then $f^{-1}(x) =$

*Solution.* $y = \frac{3x+2}{5x-3}, 5xy - 3y = 3x + 2$, Replace y with x $x(5y-3) = 3y + 2$

Hence, $x = \frac{3y+2}{5y-3}, f^{-1}(y) = \frac{3y+2}{5y-3}$, $f^{-1}(x) = \frac{3x+2}{5x-3}$ □

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**Problem 1.8.** Domain of $f(x) = \log|\log x| is$

*Solution.* $f(x) = \log|\log x|$, $f(x)$ is defined if $|\log > 0|, x > 0$ '.' $|\log x > 0|$, $x$ not equal 1 and $x > 0$. Therefore, the domain is $(0,1)U(1,\infty)$ □

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**Problem 1.9.** Given that $f(x) = \log\frac{1+x}{1-x}$ and $g(x) = \frac{3x+x^3}{1+3x^2}$, then $f(g(x)) =$ (with respct to $f(x)$)

*Solution.* $f(g(x)) = \log(\frac{1+g(x)}{1-(g(x))}) = \log(\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}) = \log(\frac{1+x}{1-x})^3 = 3f(x)$ □

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**Problem 1.10.** The domain of the function $f(x) = \sqrt{\cos^{-1}(\frac{1-|x|}{2})}$ is

*Solution.* $\cos^{-1}(\frac{1-|x|}{2})$ is defined if $-1 \leq \frac{1-|x|}{2} \leq 1,, -3 \leq -|x| \leq 1,, -1 \leq |x| \leq 3,,, |x| \geq -1$

is true for all real values of $x$ Hence, $|x| \leq 3,, -3 \leq x \leq 3$ Also $\cos^{-1}(\frac{1-|x|}{2}) \geq 0$

For all $x[-3,3]$. Therefore the domain is $[-3,3]$ □

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**Problem 1.11.** The value of $\frac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta}$ is

*Solution.* $\frac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta} = \frac{2\cos^2\theta+2\sin\theta\cos\theta}{2\sin^2\theta+2\sin\theta\cos\theta} = \frac{2\cos\theta(\cos\theta+\sin\theta)}{2\sin\theta(\cos\theta+\sin\theta)} = \cot\theta$ □

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**Problem 1.12.** If $\cot(\alpha+\beta) = 0$ (where $\alpha, \beta \in 1^{st}$ quadrant), then $\sin(\alpha + 2\beta) =$

*Solution.* $\cot(\alpha+\beta) = 0$ means $\alpha + \beta = 90\circ$. Therefore, $\sin(\alpha+2\beta) = \sin((\alpha+\beta)+\beta) = \sin(90+\beta) = \cos\beta$ □

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**Problem 1.13.** If $\frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta} + \cos(\alpha-\beta)\sec(\alpha+\beta) + 1^{-1} = 1$ then $\tan\alpha\tan\beta$

*Solution.* As $\frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta} + \cos(\alpha-\beta)\sec(\alpha+\beta) + 1^{-1} = 1$, then $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} * \frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)} + \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)} = 1$
$\tan\alpha\tan\beta + \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{2\cos\alpha\cos\beta} = 1 ==> \frac{1}{2}\tan\alpha\tan\beta + \frac{1}{2} = 1$ Hence, $\tan\alpha\tan\beta = 1$ □

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**Problem 1.14.** The least value of $2\log_{10}x - \log_x 0.01$ for $x > 1$ is

*Solution.* $2\log_{10}x - \log_x 0.01 = 2\log_{10}x + 2\log_x 10 = 2(t + \frac{1}{t}) \geq 4$
where $t = \log_{10}x$ and $t$ is $(+)$ as $x > 1$. □

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**Problem 1.15.** If $e^{\ln x + \log_{\sqrt{e}} x + \log_{\sqrt[3]{e}} x + \ldots\ldots \log_{\sqrt[10]{e}} x} = x^{11n}$

*Solution.* $e^{\ln x+\log_{\sqrt{e}} x+\log_{\sqrt[3]{e}} x+......\log_{\sqrt[10]{e}} x} =$
$\ln x + 2\ln x + 3\ln x + ..... + 10\ln x = (1 + 2 + 3 + .... + 10)\ln x = \frac{10*11}{2}\ln x$
$= 55\ln x = \ln x^{55}$ □

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**Problem 1.16.** The identity $\log_b n \log_a n + \log_b n \log_c n + \log_c n \log_a n$ equals

*Solution.* $\log_b n \log_a n + \log_b n \log_c n + \log_c n \log_a n = \frac{1}{\log_n b \log_n a} + \frac{1}{\log_n b \log_n c} + \frac{1}{\log_n c \log_n a} =$
$\frac{\log_n c+\log_n a+\log_n b}{\log_n a \log_n b \log_n c} = \frac{\log_n abc}{\log_n a \log_n b \log_n c}$ □

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**Problem 1.17.** If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$ then the value of $x^a y^b z^c$ is

*Solution.* $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b} = k$ Then, $\log x = k(b-c) ==> x = e^{k(b-c)}$ (i)
$\log y = k(c-a) ==> x = e^{k(c-a)}$ (ii)
$\log z = k(a-b) ==> x = e^{k(a-b)}$ (iii)
Then, $x^a y^b z^c = e^{kab-kac} * e^{kbc-kab} * e^{kac-kbc} = e^0 = 1$(iv) Also, $\log x + \log y + \log z = k(b-c) + k(c-a) + k(a-b) = 0$
$\log(xyz) = 0, xyz = 1$ (v)
from (iv) and (v) $x^a y^b z^c = xyz$ □

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**Problem 1.18.** If $1 < a \leq x$,then the minimum value of $\log_a x + \log_x a$ is

*Solution.* We have $1 < a \leq x$ ∴ $0 < \log a \leq \log x$, $\frac{\log x}{\log a} > 0$ (using arithmetic mean and geometric mean for positive number)
∴ $\frac{\log_a x+\log_x a}{2} \geq \sqrt{\frac{\log x \log a}{\log x \log a}}$
$\frac{\log x}{\log a} + \frac{\log a}{\log x} \geq 2\sqrt{\frac{\log x \log a}{\log x \log a}}$ then, $\log_a x + \log_x a \geq 2$ Hence, the minimum value of $\log_a x + \log_x a = 2$ □

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**Problem 1.19.** Find the values of $\theta$ for which the function $f(\theta) = \frac{\sin\theta}{-\sin\theta}$ is not defined

*Solution.* $f(\theta)$ is not defined when $\cos\theta - \sin\theta = 0$
$\cos\theta = \sin\theta, \tan\theta = 1, \theta = \frac{pi}{4} + n\pi$ □

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**Problem 1.20.** If $\cos p\theta = \cos q\theta$, $p$ not equal $q$, then $\theta =$

*Solution.* Given, $\cos p\theta = \cos q\theta$
$p\theta = 2n\pi(+or-)q\theta$
$(p + or - q)\theta = 2n\pi$,then $\theta = \frac{2n\pi}{(p+or-q)}$ □

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